MATH3404 Week 9

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Slides at `https://simonnthomasmaths.github.io/teaching/`

The optimal control problem

Let $\boldsymbol{x}(t) \in \mathbb{R}^n$ be a vector which describes the state of a system at time t. Let $u(t)$ be an admissible (piecewise consinuous and restricted to a bounded domain) control. We are given:

- State equations: $\dot{x}_i = f_i(\boldsymbol{x}, u)$, where $i = 1, 2, \ldots, n$.
- An initial state $\boldsymbol{x}(t_0) = \boldsymbol{x}^0$ at a given time t_0.
- A target state $\boldsymbol{x}(t_1) = \boldsymbol{x}^1$ which should be reached at time t_1.
- A cost functional

$$J = \int_{t_0}^{t_1} f_0(\boldsymbol{x}, u)dt$$

Our aim is to find an *optimal control* $u^*(t)$ with a corresponding path $\boldsymbol{x}^*(t)$ which transfers the system from $\boldsymbol{x}^0$ to $\boldsymbol{x}^1$ in such a way that J is minimised.

Pontryagin maximum principle

Theorem

Let $u^*(t)$ be an admissible control with path $\mathbf{x}^* = (x_1^*, x_2^*)$ that transfers the system from $\mathbf{x}^0$ at time $t = t_0$ to x^1 at an unspecified time t_1. If u^* and $\mathbf{x}^*$ are optimal (minimize J), then there exists a non-trivial $\boldsymbol{\psi} = (\psi_0, \psi_1, \psi_2)^T$ satisfying

$$\dot{\psi}_i = -\frac{\partial H}{\partial x_i}, \quad i = 0, 1, 2, \tag{1}$$

and a function

$$H(\boldsymbol{\psi}, \mathbf{x}, u) = \psi_0 f_0(\mathbf{x}, u) + \psi_1 f_1(\mathbf{x}, u) + \psi_2 f_2(\mathbf{x}, u)$$

such that

- *For all $t \in [t_0, t_1]$, H attains its maximum with respect to u at $u = u^*(t)$,*
- *$H(\boldsymbol{\psi}^*, \mathbf{x}^*, u^*) = 0$ and $\psi_0 \leq 0$ at $t = t_1$, where $\boldsymbol{\psi}^*(t)$ is the solution of (1) for $u = u^*(t)$.*

Here, x_0 is defined to be the solution of the differential equation

$$\dot{x}_0 = f_0(x_1, x_2, u)$$

with initial condition $x_0(t_0)$.

Reading recommendation

This material is challenging when you first encounter it. I recommend reading through chapter 4 of Optimal Control and the Calculus of Variations, by Enid R. Pinch.