MATH3404 Week 8

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Weierstrass condition

Let $x^*(t)$ be an extremal for $J[x]$ in a field of extremals with slope function $p(x,t)$. Let $\mathcal{C}, x = x(t)$ be any other curve joining the endpoints. Define the Weierstrass excess function to be

$$E(t,x,\dot{x},p) := f(t,x,\dot{x}) - f(t,x,p) - (\dot{x} - p)\frac{\partial f(t,x,p)}{\partial p}.$$

Theorem (The Weierstrass condition)

Suppose x^* is an extremal of $J[x]$. If

- *x^* is a member of a field of extremals with slope function $p(t,x)$, and*
- *$E(t,x,\dot{x},p) \geq 0$ for all points (t,x) "sufficiently close" to x^*, and any $\dot{x}$,*

then x^* is a strong local minimum for $J[x]$.

The choice of field of extremals is not important. We only care whether one exists.

Differential equations trick

Suppose we have a linear homogeneous second order DE:

$$a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = 0. \qquad (1)$$

In general, there is no closed formula for $x(t)$. We do know, however, that if $y_1(t)$ and $y_2(t)$ are *linearly independent* solutions, then the general solution is

$$x(t) = Ay_1(t) + By_2(t).$$

Differential equations trick

Suppose we have a linear homogeneous second order DE:

$$a(t)\ddot{x}(t) + b(t)\dot{x}(t) + c(t)x(t) = 0. \tag{1}$$

Theorem

Let y_1, y_2 be solutions of a linear homogeneous second order DE. If there exists $t \in I$ such that

$$\mathrm{Wr}(y_1, y_2)(t) = y_1(t)\dot{y}_2(t) - y_2(t)\dot{y}_1(t) \neq 0,$$

then y_1 and y_2 are linearly independent on I.

Thus, if

$$y_1(t_0) = 0, \quad \dot{y}_1(t_0) = 1, \quad y_2(t_0) = 1, \quad \dot{y}_2(t_0) = 0,$$

then the general solution for (1) is

$$x(t) = Ay_1(t) + By_2(t).$$