MATH3404 Week 7

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Transversality condition

From Pinch (page 41).

Problem

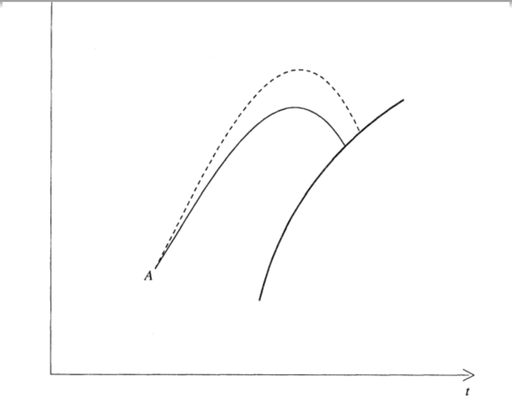
Minimize

$$J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt$$

when $(t_0, x(t_0))$ *is fixed, but* $(t_1, x(t_1))$ *is required to lie on a curve* $x = c(t)$.

- Euler-lagrange equation gives us an extremal $x^*(t)$ with two unknowns, so we need two conditions to find them.
- We can use the fixed endpoint to find one unknown.
- We use the transversality condition to find the other.

Transversality condition



- Transversality condition:

$$f(t_1) + (\dot{c}(t_1) - \dot{x}^*(t_1))\frac{\partial f}{\partial \dot{x}}(t_1) = 0$$

- Gives us conditions on $\dot{x}^*(t)$ (slope of extremal), such that moving the endpoint does not decrease $J[x]$.

Field of extremals

Problem

Minimize

$$J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1.$$

We find an extremal $x^*(t)$ using the Euler-Lagrange equation.

Definition

In this context, a *field of extremals* is a family of curves, containing $x^*(t)$, which give a "simple cover" of a neighbourhood of $x^*(t)$ in the t-x plane.

In other words, each point of the plane (in a region around the extremal) is hit by exactly one member of the field of extremals.

Field of extremals

For example, if we found $x(t) = t$ (general equation $x(t) = kt + l$), then a field of extremals could be $x(t) = t + l$, or $x(t) = kt$, where $k, t \in \mathbb{R}$.

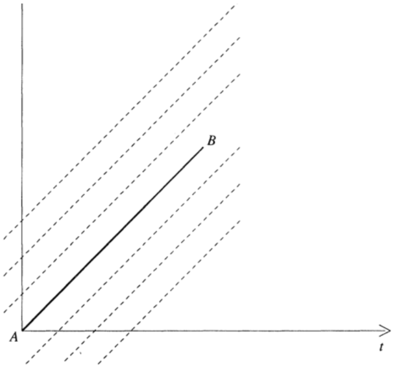


Figure 3.10 The field of extremals $x = t + l$

The slope function is defined to be $p(t, x) = \dot{x}(t)$.

Weierstrass condition

Let $x^*(t)$ be an extremal for $J[x]$ in a field of extremals with slope function $p(x,t)$. Let $\mathcal{C}, x = x(t)$ be any other curve joining the endpoints. Define the Weierstrass excess function to be

$$E(t,x,\dot{x},p) := f(t,x,\dot{x}) - f(t,x,p) - (\dot{x}-p)\frac{\partial f(t,x,p)}{\partial p}.$$

Theorem (The Weierstrass condition)

Suppose x^* is an extremal of $J[x]$. If

- *x^* is a member of a field of extremals with slope function $p(t,x)$, and*
- *$E(t,x,\dot{x},p) \geq 0$ for all points (t,x) "sufficiently close" to x^*, and any $\dot{x}$,*

then x^* is a strong local minimum for $J[x]$.