MATH3404 Week 4

Simon Thomas

September 14, 2025

Clarification

Definition

A curve $x^* : [t_0, t_1] \to \mathbb{R}$ is a *weakly local minimiser* of a functional J if there exists $\epsilon_1, \epsilon_2 > 0$ such that

$$J(x^*) \leq J(y)$$

for all y satisfying the boundary conditions and satisfying

$$|x^*(t) - y(t)| < \epsilon_1, \qquad |\dot{x}^*(t) - \dot{y}(t)| < \epsilon_2,$$

for all $t_0 \leq t \leq t_1$

Definition

A curve $x^* : [t_0, t_1] \to \mathbb{R}$ is a *strongly local minimiser* of a functional J if there exists $\epsilon > 0$ such that

$$J(x^*) \leq J(y)$$

for all y satisfying the boundary conditions and satisfying

$$|x^*(t) - y(t)| < \epsilon$$

for all $t_0 \leq t \leq t_1$

So if x^* is a strongly local minimiser of J, then $J(x^*) \leq J(y)$ for a broader set of y than is required to be a weakly local minimiser. Therefore,

$$x^* \text{ is a strongly local minimiser} \quad \Longrightarrow \quad x^* \text{ is a weakly local minimiser}$$

Extremals

Given a fixed endpoint problem

$$J[x] = \int_{t_0}^{t_1} f(t, x, \dot{x})dt, \quad x(t_0) = x_0, \quad x(t_1) = x_1,$$

the *extremels* are the solutions x to the Euler-Lagrange equation,

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) = 0.$$

Integrand with no t

If $f(t, x, \dot{x})$ does not depend explicitly on t, then Euler-Lagrange implies that the extremels are the solutions to

$$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = A,$$

for some constant $A \in R$.
For example,

$$f(t, x, \dot{x}) = \sin(t)x^2 + \frac{t}{\dot{x}}$$

depends on t explicitly, while

$$f(t, x, \dot{x}) = x^2 + \frac{x}{\dot{x}}$$

does not.