MATH3404 Week 3

Simon Thomas

September 14, 2025

Example

If

$$H(a) = \begin{pmatrix} b & c & d \\ e & f & g \\ h & i & j \end{pmatrix},$$

then the leading principal minors are

$$\det \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad \det\left(b\right) = b.$$

Therefore, if $\det\left(b\right) < 0$, $\det \begin{pmatrix} b & c \\ e & f \end{pmatrix} > 0$, and $\det H(a) < 0$, then a is a local maximum.

Problem

Find local minima and maxima of $f(x)$ in $\mathbb{R}^n$ such that x satisfies

$$g_j(x) = c_j, \qquad j = 1, \ldots, m < n.$$

Find critical points

Find $x \in \mathbb{R}^n$ and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$ such that

$$\nabla L(x) = 0$$

and

$$g_j(x) = c_j,$$

where $L(x) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x)$.

Check whether critical points are degenerate

Let H_L be the Hessian of L and define

$$B = \nabla g = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}.$$

The bordered Hessian is

$$H = \begin{pmatrix} H_L & B \\ B^T & 0 \end{pmatrix}.$$

If $\det H \neq 0$ at a critical point a, then a is a nondegenerate critical point.

Minimum vs Maximum

Let a be a nondegenerate critical point. We say that h is a tangent vector if $h^T \nabla g = 0$.

- If
$$h^T H_L h \geq 0$$
for all tangent vectors at a, then a is a local minimum.
- If
$$h^T H_L h \leq 0$$
for all tangent vectors at a, then a is a local maximum.