MATH3404 Week 2

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September 14, 2025

Finding local minima/maxima

Let $f : \mathbb{R}^n \to \mathbb{R}$.

Example

- $f(x) = x^2$
- $f(x_1, x_2, x_3) = \sin(x_1 x_3) + 5x_3$

Question

How can we find any local minima or maxima?

Case 1: $n = 1$

1. Find critical points i.e. find $x \in \mathbb{R}$ where
   - f is not differentiable at x, or
   - $f'(x) = 0$.
2. If f is C^2 at x (continuous second derivatives),
   - if $f''(x) > 0$, then x is a local min,
   - if $f''(x) < 0$, then x is a local max,
   - if $f''(x) = 0$, inconclusive. Use Theorem 1.2 or Theorem 1.3.
3. If f is not C^2 at x, use the definitions of local maximum/minimum to decide.

Case 2: $n > 1$

1. Find $a \in \mathbb{R}^n$ such that $\nabla f(a) = 0$, where
$$\nabla f := \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix}$$

2. Compute Hessian matrix
$$H = \begin{pmatrix} \dfrac{\partial^2 f}{\partial x_1^2} & \dfrac{\partial^2 f}{\partial x_1 \, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_1 \, \partial x_n} \\ \dfrac{\partial^2 f}{\partial x_2 \, \partial x_1} & \dfrac{\partial^2 f}{\partial x_2^2} & \cdots & \dfrac{\partial^2 f}{\partial x_2 \, \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial^2 f}{\partial x_n \, \partial x_1} & \dfrac{\partial^2 f}{\partial x_n \, \partial x_2} & \cdots & \dfrac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

- If $h^T H(a) h > 0$ for all $h \in \mathbb{R}^n$, then a is a local minimum.
- If $h^T H(a) h < 0$ for all $h \in \mathbb{R}^n$, then a is a local maximum.

$h^T H(a) h > 0$ for all $h \in \mathbb{R}^n$ $\iff$ All eigenvalues of $H(a)$ are positive

$\iff$ $\det H(a) > 0$ and all leading principal minors of $H(a)$ are positive

$h^T H(a) h < 0$ for all $h \in \mathbb{R}^n$ $\iff$ All eigenvalues of $H(a)$ are negative

$\iff$ $(-1)^n \det H > 0$ and leading principal minors alternate in sign with $\dfrac{\partial^2 f}{\partial x_1^2} < 0$.

Example

If

$$H(a) = \begin{pmatrix} b & c & d \\ e & f & g \\ h & i & j \end{pmatrix},$$

then the leading principal minors are

$$\det \begin{pmatrix} b & c \\ e & f \end{pmatrix}, \quad \det\left(b\right) = b.$$

Therefore, if $\det\left(b\right) < 0$, $\det \begin{pmatrix} b & c \\ e & f \end{pmatrix} > 0$, and $\det H(a) < 0$, then a is a local maximum.