MATH3404 Week 11

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Slides at `https://simonnthomasmaths.github.io/teaching/`

Time optimal control

Problem

The system

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + u\boldsymbol{l}$$

is to be controlled to the origin in minimum time subject to $|u| \leq 1$.

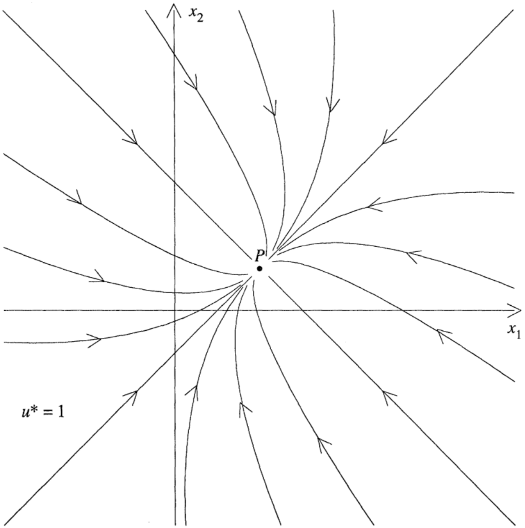
If $\det A \neq 0$, then the trajectories associated with u^* are given by

$$\boldsymbol{x}(t) = \alpha \boldsymbol{v}_1 e^{\lambda_1 t} + \beta \boldsymbol{v}_2 e^{\lambda_2 t} + \boldsymbol{x}_s,$$

where $\boldsymbol{x}_s$ is the isolated singularity for u^*, λ_i are the eigenvalues of A, and $\boldsymbol{v}_i$ are the corresponding eigenvectors.
The direction of travel along the trajectories depends on the eigenvalues λ_i.

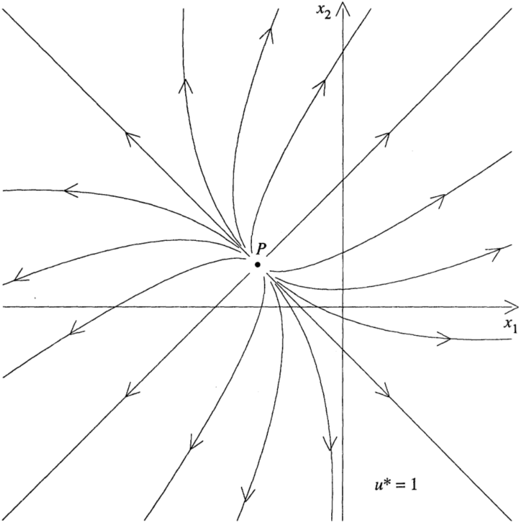
Stable node

If $\lambda_1 < 0$ and $\lambda_2 < 0$ then we have *stable nodes* at P and Q.



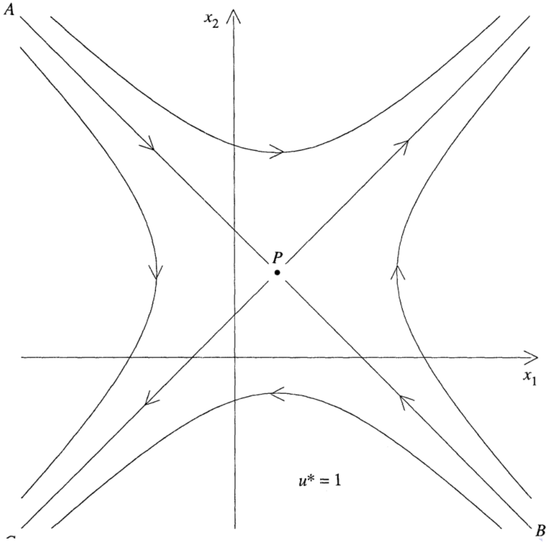
Unstable node

If $\lambda_1 > 0$ and $\lambda_2 > 0$ then we have *unstable nodes* at P and Q.



Saddle point

If one eigenvalue is positive and one negative then we have *saddle points* at P and Q.



Cost depends on final position

Problem

The system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, u)$ *is to be controlled, during the fixed time interval* $t_0 \leq t \leq t_1$, *from a given initial state* $\boldsymbol{x}^0$ *in such a way that*

$$J = g(\boldsymbol{x}(t_1)) + \int_{t_0}^{t_1} f_0(\boldsymbol{x}, u)dt$$

is minimized. Find the optimal control.

Approach: Maximise

$$H = -f_0 + \sum_{j=1}^{n} \lambda_j f_j,$$

as a function of u, where $\dot{\lambda}_i = -\frac{\partial H}{\partial x_i}$ and using the end conditions $\boldsymbol{x}(t_0) = \boldsymbol{x}^0$ and $\lambda_i(t_i) = -\frac{\partial g}{\partial x_i}(t_i)$.