MATH3404 Week 10

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Slides at `https://simonnthomasmaths.github.io/teaching/`

Optimal control general recipe

- Find $f_0(\boldsymbol{x}, u)$ using $J = \int_{t_0}^{t_1} f_0(\boldsymbol{x}, u)dt$.
- Find ψ_i using $\dot{\psi}_i = -\frac{\partial H}{\partial x_i}$.
- Maximise H as a function of u to find u^*.
- Find trajectory $\boldsymbol{x}(t)$ associated to u^*.
- Determine path from $\boldsymbol{x}^0$ to $\boldsymbol{x}^1$ using these trajectories.

Consider a system with two variables $x_1(t)$, $x_2(t)$ and a single control variable $u(t)$ restricted to $|u| \leq 1$. The system is governed by

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + u\boldsymbol{l},$$

where

$$\boldsymbol{l} = \begin{pmatrix} l \\ m \end{pmatrix}.$$

We want to find the optimal control $u^*(t)$ for which

$$J = \int_{t_0}^{t_1} 1 dt = t_1 - t_0$$

is minimised.

Time-optimal control of linear systems

- Optimal control is of the form

$$u^* = \operatorname{sgn}(l\psi_1 + m\psi_2).$$

- Provided $\det A \neq 0$, the trajectories for u^* will have isolated singularities (solutions constant for all t) when

$$A\boldsymbol{x} + u^*\boldsymbol{l} = 0.$$

- If the eigenvalues of A are real, then u^* switches at most once between -1 and 1.
- The trajectories associated with u^* are given by

$$\boldsymbol{x}(t) = \alpha \boldsymbol{v}_1 e^{\lambda_1 t} + \beta \boldsymbol{v}_2 e^{\lambda_2 t} + \boldsymbol{x}_s,$$

where $\boldsymbol{x}_s$ is the isolated singularity for u^*, λ_i are the eigenvalues of A, and $\boldsymbol{v}_i$ are the eigenvectors.